MATH 147 - Fall 2023

Midterm practice problems

For the exam you must know the following definitions and main theorems/facts (and be able to reproduce their proofs):

Definitions: ε-N definition of limit of the sequence, convergent/divergent sequences; definition of infinite limits (when the sequence diverges to $\pm\infty$); definition of supremum/infimum; definition of monotone sequence; definition of subsequence; definition of Cauchy sequence; definition of the limit of function (one-sided limits, limits at infinity, infinite limits).

Theorems/Facts: Principle of Mathematical Induction; Well-Ordering Principle; Squeeze theorem; Least Upper Bound Principle (and Greatest Lower Bound Principle with proof using LUBP); Monotone Convergence Theorem (and proof); Bolzano-Weierstrass theorem (and proof); any Cauchy sequence is convergent (and proof); Sequential characterization for limits (and proof); limit properties and rules for sequences and functions (and proofs).

This is just a rough list of main topics/theorems covered in class that can serve you as a guide in preparation for the midterm. There are other facts that were covered in class that I did not list here but which you are also expected to know (or be able to reproduce their proofs).

Problem 1. Verify the following limits using ε-N definition of limit of the sequence:

(a) $\lim\limits_{n\to\infty} \frac{n+3}{n^2+4} = 0$.

(b) $\lim\limits_{n\to\infty} \frac{\sqrt{64n^4+3n^3+8}}{n^2} = 8$.

Problem 2. Prove that $\lim a_n = 0$ if and only if $\lim |a_n| = 0$. Give an example that, in general, convergence of $\{|a_n|\}$ does not imply convergence of $\{a_n\}$.

Problem 3. Show that if $a_n > 0$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} a_n = 0$ if and only if $\lim\limits_{n\to\infty} \frac{1}{a_n} = +\infty$.

Problem 4. Suppose that $a_n \geq 0$ and $\{a_n\}$ is convergent, that is, $\lim\limits_{n\to\infty} a_n = L$. Prove that $\lim\limits_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

Problem 5. Let $S \subseteq \mathbb{R}$ be a non-empty subset that is bounded below. Prove that

$$\inf S = -\sup(-S)$$

where $-S := \{-s : s \in S\}$.

Problem 6. Consider the set $S = \{(-1)^n \left(1 - \frac{1}{n}\right) : n \in \mathbb{N}\} \subseteq \mathbb{R}$.

(a) Show that 1 is an upper bound for S.

(b) Show that if α is an upper bound for S, then $\alpha \geq 1$. (*Hint: proof by contradiction might be helpful*)

(c) Conclude that $\sup S = 1$.

Problem 7. Let $a_1 \geq 2$ and $a_{n+1} = 1 + \sqrt{a_n - 1}$ for $n \in \mathbb{N}$. Show that a_n is decreasing and bounded below by 2. Find the limit.

Problem 8.

(a) Let $\{a_n\}$ and $\{b_n\}$ are convergent sequences. Show that if $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

(b) Let $\{a_n\}$ be an increasing sequence, $\{b_n\}$ be a decreasing sequence, and suppose that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Problem 9. Let $0 < r < 1$. If $|a_{n+1} - a_n| \leq r^n$ for all $n \in \mathbb{N}$, show that $\{a_n\}$ is a convergent sequence.

Problem 10. Prove that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^k}{k}$ exists and is finite.

Problem 11. Let $a_1, \ldots, a_n \in \mathbb{R}$. Show the following inequalities:

(a)

$$\left(\frac{1}{n} \sum_{i=1}^{n} a_i \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} a_i^2.$$

(b) Assume that $a_1, \ldots, a_n \in \mathbb{R}$ are all positive.

$$\left(\sum_{i=1}^{n} a_i \right) \left(\sum_{i=1}^{n} \frac{1}{a_i} \right) \geq n^2.$$

Problem 12. Using ε-δ definition of limit of the function, determine whether the following limits exist. If yes, find the limit, if no, justify your answer.

(a) $\lim_{x\to 2}(x^2 - 2x + 1)$.

(b) $\lim_{x\to 4} \frac{|x^2-2x-8|}{x-4}$.

Problem 13. Let f and g be real-valued functions defined on $(a, b) \subset \mathbb{R}$ such that $c \in (a, b)$.

(i) Show that if both limits $\lim_{x\to c} f$ and $\lim_{x\to c} f + g$ exist, then $\lim_{x\to c} g$ exist.

(ii) If $\lim_{x\to c} f$ and $\lim_{x\to c} fg$ exist, does it follow that $\lim_{x\to c} g$ exist?

Problem 14. Let $f : (a, b) \to \mathbb{R}$ such that $c \in (a, b)$. Assume that $\lim_{x\to c} f$ exists. Prove that $\lim_{x\to c} |f| = |\lim_{x\to c} f|$ where $|f|(x) = |f(x)|$ for all $x \in (a, b)$.

Problem 15. Compute the following limits or show that they do not exist (show your work!):

(a) $\lim_{n\to\infty} \frac{n}{n+1} - \frac{n+1}{2n}$

(b) $\lim_{n\to\infty} \sqrt{n^2-3n} - \sqrt{n^2+n+1}$

(c) $\lim_{n\to\infty} \frac{\sin(n^2+1)}{n}$

(d) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

(e) $\lim_{x\to 2^-} \frac{x^3-x^2-4}{x^2-4}$.