MATH 147 - Fall 2023

Assignment 2

(due Thursday, October 5 at 5 pm ET on Crowdmark)

Problem 1. (15 points) It seems obvious that $\mathbb{N}$ is not bounded in $\mathbb{R}$ but how we can prove this? The absence of upper bounds of $\mathbb{N}$ means that for any real number $x \in \mathbb{R}$ there exists a natural number $n_x \in \mathbb{N}$ (depending on x) such that $x < n_x$. This is the so-called Archimedean Property:

Archimedean Property: *If $x \in \mathbb{R}$ then there exists $n_x \in \mathbb{N}$ such that $x < n_x$.*

(o) Read and understand the proof of Archimedean Property in the notes *(click to open the file in the browser)* on page 77.

(i) Prove that $\inf S = 0$ where $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Now we can state the so-called Archimedean Property II:

Archimedean Property II: *If $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $0 < \frac{1}{n_\varepsilon} < \varepsilon$.*

Proof: By part *(i)*, we have that $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$. Since $\varepsilon > 0$, then ε cannot be a lower bound for the set $\{\frac{1}{n} : n \in \mathbb{N}\}$. Hence, there exists $n_\varepsilon \in \mathbb{N}$ such that $0 < \frac{1}{n_\varepsilon} < \varepsilon$.

(ii) Show that for $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon - 1 \leq \varepsilon \leq n_\varepsilon$.

(iii) Prove that for any real numbers x and y with $x < y$, there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

Hint: *Assume that $x > 0$ (why can we do this?).*

Part *(iii)* says that given any two real numbers, there is always a rational number between them. We say that the set of rational numbers is *dense* in $\mathbb{R}$.

Problem 2. (8 points)

(i) Let $\{a_n\}$ be a bounded sequence of real numbers and let $s = \sup\{a_n : n \in \mathbb{N}\}$. Show that if $s \notin \{a_n : n \in \mathbb{N}\}$ then there is a subsequence of $\{a_n\}$ that converges to s.

(ii) Give an example of a sequence in $\mathbb{R}$ which has three subsequences converging to -2, 0 and 3, respectively.

Problem 3. (15 points)

We say that a sequence $\{a_n\}$ of real numbers is *contractive* if there exists a constant C with $0 < C < 1$ such that

$$|a_{n+2} - a_{n+1}| \leq C|a_{n+1} - a_n|$$

for all $n \in \mathbb{N}$.

(i) Show that every contractive sequence is a Cauchy sequence. (and hence, convergent)

(ii) Let $f_1 = f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$ be the Fibonacci sequence. The first few terms of the sequence are $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$. Now define

$$a_n = \frac{f_n}{f_{n+1}}.$$

Show that $\{a_n\}$ satisfies the recursive relation $a_1 = 1$ and $a_n = \frac{1}{1+a_{n-1}}$ for all $n \geq 2$, and then prove that $\frac{1}{2} \leq a_n \leq 1$.

(iii) Show that $\{a_n\}$ is contractive, and find all possible values of L such that $a_n \to L$.

Problem 4. (15 points) Given a bounded sequence $\{x_n\}$, we define the lim sup and lim inf of $\{x_n\}$ by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$$

and

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}.$$

(i) Prove that the sequences $\{s_n\}$ and $\{t_n\}$ converge where $s_n = \sup\{x_k : k \geq n\}$ and $t_n = \inf\{x_k : k \geq n\}$.

(ii) Find lim sup and lim inf of the sequence $a_n = \frac{3(-1)^n n^2}{n^2-n+1}$.

(iii) Show that if $\{a_n\}$ and $\{b_n\}$ are bounded sequences, then

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

Give an example in which two sides are not equal.

Problem 5. (15 points) In literature, one might see an infinite series defined as an expression of the form:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots.$$

However, it is not clear what value can be attached to this formal expression.

Definition: Let $\{a_n\}$ be a sequence of real numbers. For each $n \in \mathbb{N}$, we define the *partial sum S_n of order n (or n-th partial sum)* by

$$S_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $\{S_n\}$ converges to some $S \in \mathbb{R}$. In this case, we write $\sum_{n=1}^{\infty} a_n = S$ and say that S is the *sum* (or the *value*) of the series $\sum_{n=1}^{\infty} a_n$. Otherwise, we say that $\sum_{n=1}^{\infty} a_n$ diverges.

(i) Prove that if the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(ii) Determine whether the series $\sum_{n=1}^{\infty} \cos(\pi n)$ converges or diverges. If it converges, find the limit, if not, justify your answer.

(iii) Let $r \in \mathbb{R}$ be such that $|r| < 1$. The series $\sum_{n=0}^{\infty} r^n$ is called the geometric series.

Prove that $S_n = \frac{1-r^{n+1}}{1-r}$, conclude that $\sum_{n=0}^{\infty} r^n$ converges and find the sum of the geometric series.

Problem 6. (10 points) Using either ε-δ definition or sequential characterization for limits determine whether the following limits exist. If yes, find the limits, if not, justify your answer.

(i) $\lim_{x \to 1} \frac{x^2-x+1}{x+1}$.

(ii) $\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)$.