Assignment 2

1.

i)

Greatest Lower Bound

Let $S \subseteq \mathbb{R}$. Then β is called the *greatest lower bound* of S if

1. β is a lower bound of S.
2. β is the largest such lower bound. That is, if $\gamma \leq x$ for every $x \in S$, then $\gamma \leq \beta$.

We write

$$\beta = glb(S).$$

Note: The greatest lower bound is often called the *infimum* of S and is denoted by

$$inf(S).$$

Here $\beta = 0$, $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$

1. $0 \leqslant s \quad , \forall s \in S$

$\Leftrightarrow 0 \leqslant \frac{1}{n} \quad , \forall n \in \mathbb{N}$ → trivial

Since $n \geqslant 0$, we can multiply both sides of this inequality with n and it will still hold

$\Leftrightarrow 0 \leqslant 1$

2. We will take the contrapositive and prove it by contradiction.

$\gamma > 0 \Rightarrow \exists s \in S : \gamma < s$ ← Contrapositive of 2.

SFAC $\nexists s \in S : \gamma < s$

$\gamma \in \mathbb{R} \Leftrightarrow \frac{1}{\gamma} \in \mathbb{R}$

By Archimidean Property $\exists n \in \mathbb{N} : n > \frac{1}{\gamma}$

$\Rightarrow \frac{1}{n} < \gamma$ Same thing has been given below as Archimidean Property II (in H.W.)

Which forms a contradiction $\therefore \exists s \in S : \gamma < s$

$\therefore \inf S = 0$

ii) Show that for $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon - 1 \leq \varepsilon \leq n_\varepsilon$.

The smallest value of $n_\varepsilon \in \mathbb{N} = 1$

$\therefore$ smallest value of $n_\varepsilon - 1$, $n_\varepsilon \in \mathbb{N} = 0$

$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \quad n_\varepsilon - 1 \leqslant \varepsilon$ trivial since lower bound of $n_\varepsilon - 1$ is 0

If ε is an Natural Number, we can take $n_\varepsilon = \varepsilon$

Since $n_\varepsilon - 1 < n_\varepsilon$

$\therefore n_\varepsilon - 1 < \varepsilon < n_\varepsilon$ is obviously true

By the Archimidean Property,

$\varepsilon \in \mathbb{R} \Rightarrow \exists n_\varepsilon \in \mathbb{N} : \varepsilon < n_\varepsilon$

Let $S = \{ n_\varepsilon \in \mathbb{N} : n_\varepsilon > \varepsilon \}$

Since $S \subseteq \mathbb{N}$, S must be well ordered

Let the smallest element of S be s_0

$\Rightarrow s_0 - 1 < \varepsilon$

if not $s_0 - 1 < \varepsilon$. This forms a contradiction
($s_0 = 1$ is trivial and has been proved above)

$\therefore \forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} : n_\varepsilon - 1 \leqslant \varepsilon \leqslant n_\varepsilon$

iii) Prove that for any real numbers x and y with $x < y$, there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

We can assume $x > 0$

↳ if $x \leqslant 0$ there are 2 possibilities

$y > 0$, then $r = 0$ is a trivial solution

$y \leqslant 0$, then we can multiply all sides of the inequality by -1 , and we get

$-y < -r < -x$ where $-x, -y, -r > 0$

We can take new $x_0 = -y$, $y_0 = -x$, $r_0 = -r$

$x_0 < r_0 < y_0$

Showing $\exists\, r_0 \in \mathbb{Q}$ proves $\exists\, r \in \mathbb{Q}$: $x < r < y$

$x > 0 \Rightarrow r > 0$

$\therefore$ we can write r as $\frac{a}{b}$ where $a, b \in \mathbb{N}$

$$x < \frac{a}{b} < y$$

since $x < y$, Let $y = x + \varepsilon$, $\varepsilon > 0$

$$x < \frac{a}{b} < y \Rightarrow bx < a < b(x+\varepsilon)$$

$bx < a$ is arbitrary because of the Archimidean Property

$\exists\, b \in \mathbb{N}$: $b \cdot \varepsilon > 2 \Leftrightarrow b > \frac{2}{\varepsilon}$

Due to the Archimidean Property, we know there exists a b that satisfies this.

$$bx < a < bx + 2$$

We know there exists an $a = \lfloor b \cdot x \rfloor + 1$ that satisfies this.

2.

i) Let $\{a_n\}$ be a bounded sequence of real numbers and let $s = \sup\{a_n : n \in \mathbb{N}\}$. Show that if $s \notin \{a_n : n \in \mathbb{N}\}$ then there is a subsequence of $\{a_n\}$ that converges to s.

Let $\mathbb{B}$ be a subsequence of $\{a_n\}$ such that

$\mathbb{B} = \{a_k, k \in \mathbb{N} \text{ s.t. } a_k > a_n, \forall n \in \mathbb{N} \text{ s.t. } n < k\}$

$\mathbb{B}$ isn't empty a_1 belongs to it (trivial)

$\mathbb{B}$ is infinite. We can prove this using contradiction

If $\mathbb{B}$ is finite, let its largest element be b_0

$b_0 \in \{a_n : n \in \mathbb{N}\}$ (trivial since it's a subsequence)

If $\mathbb{B}$ is finite $\Rightarrow$ $\nexists a_n, n \in \mathbb{N}$ s.t. $a_n > b_0$

This would mean $\sup\{a_n\} = b_0$, but that contradicts

$s \notin \{a_n\}$. $\therefore$ $\mathbb{B}$ is infinite.

$\mathbb{B}$ is non decreasing (trivial the n_{th} element by definition is larger than the elements before it)

$\therefore$ By MCT

$\mathbb{B}$ converges to $\sup\{\mathbb{B}\} = s$

$\sup\{\mathbb{B}\} = s$ (trivial since $\mathbb{B}$ essentially contains all largest elements of a_n upto that point)

ii) Take sequence $-2, 0, 3, 2, 0, -3 \dots$

where $a_n = -2$ if $n \equiv 1 \pmod 3$
$a_n = 0$ if $n \equiv 2 \pmod 3$
$a_n = 3$ if $n \equiv 0 \pmod 3$

The subsequences $\mathbb{B}$, $\mathbb{M}$ and $\mathbb{P}$ as defined below converge to $-2, 0, 3$ respectively

$$\mathbb{B} = \{a_{3k+1} : k \in \mathbb{N}\}$$

$$\mathbb{M} = \{a_{3k+2} : k \in \mathbb{N}\}$$

$$\mathbb{P} = \{a_{3k} : k \in \mathbb{N}\}$$

3.

i) Show that every contractive sequence is a Cauchy sequence. (and hence, convergent)

Cauchy Sequence

We say that a sequence $\{a_n\}$ is *Cauchy* if for every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|a_n - a_m| < \epsilon$.

$$|a_{n+2} - a_{n+1}| \leqslant C|a_{n+1} - a_n| \leqslant C^2|a_n - a_{n-1}| \leqslant \cdots$$

WLOG Let $n > m$

$$|a_n - a_m| = |a_n - a_{n-1} + a_{n-1} - a_{n-2} \cdots a_{m+1} - a_m|$$

By Δ ineq.

$$|a_n - a_m| \leqslant |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots |a_{m+1} - a_m|$$

$$\leqslant C^{n-2}|a_2 - a_1| + C^{n-1}|a_2 - a_1| + \cdots C^{m-1}|a_2 - a_1|$$

$$= C^{m-1}\left(\frac{1 - C^{m-n}}{1 - C}\right)|a_2 - a_1|$$

$$\leqslant C^{m-1}\left(\frac{1}{1-C}\right)|a_2 - a_1| \qquad (\text{Since } 0 < C^{m-n} < 1)$$

Let p be a constant $= \dfrac{|a_2 - a_1|}{1 - C}$

$= \beta \cdot C^{m-1}$

Since $\underset{m\to\infty}{\mathcal{L}t}\ C^{m-1} = 0$ (trivial since $0 < c < 1$)

$\forall\ \varepsilon > 0 \quad \exists\ m \in \mathbb{N} : \quad \beta \cdot C^{m-1} < \varepsilon$ (take $N > \log_c \frac{\varepsilon}{\beta} + 1$)

$\hookrightarrow m \geq N$

$\therefore$ Every contractive seq. is Cauchy.

ii) Let $f_1 = f_2 = 1$, and $f_n = f_{n-1} + f_{n-2},\ n \geq 3$ be the Fibonacci sequence. The first few terms of the sequence are $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$. Now define

$$a_n = \frac{f_n}{f_{n+1}}.$$

Show that $\{a_n\}$ satisfies the recursive relation $a_1 = 1$ and $a_n = \frac{1}{1+a_{n-1}}$ for all $n \geq 2$, and then prove that $\frac{1}{2} \leq a_n \leq 1$.

We will prove this relation using induction.

Base Case $\quad a_1 = 1 \wedge \frac{f_1}{f_2} = \frac{1}{1} = 1 \quad \therefore a_1 = \frac{f_1}{f_2}$

Let $\quad a_k = \frac{f_k}{f_{k+1}}$

$$a_{k+1} = \frac{1}{1 + a_k} = \frac{1}{1 + \frac{f_k}{f_{k+1}}} = \frac{f_{k+1}}{f_k + f_{k+1}} = \frac{f_{k+1}}{f_{k+2}}$$

$\therefore$ By Induction $\quad a_n = \frac{f_n}{f_{n+1}}$

To prove that $\frac{1}{2} \leqslant a_n \leqslant 1$

$a_1 = 1$, $a_2 = \frac{1}{2}$ (Base case)

if $\frac{1}{2} \leqslant a_k \leqslant 1$

$a_{k+1} = \frac{1}{1+a_k}$, since $\frac{1}{2} \leqslant a_k \leqslant 1$

$\Rightarrow \frac{3}{2} \leqslant a_k + 1 \leqslant 2$

$\frac{1}{2} \leqslant a_{k+1} \leqslant 1$

$\Rightarrow \frac{1}{2} \leqslant \frac{1}{a_k+1} \leqslant \frac{2}{3} \leqslant 1$

Hence by Induction $\frac{1}{2} \leqslant a_n \leqslant 1 \quad \forall n \in \mathbb{N}$

iii) Show that $\{a_n\}$ is contractive, and find all possible values of L such that $a_n \to L$.

$$|a_{n+1} - a_n| = \left| \frac{1}{1+a_n} - \frac{1}{1+a_{n-1}} \right| = \left| \frac{a_{n-1} - a_n}{(1+a_n)(1+a_{n-1})} \right|$$

$$= \frac{|a_n - a_{n-1}|}{(1+a_n)(1+a_{n-1})}$$

Since $(1+a_n), (1+a_{n-1}) > 0$ and $|-x| = |x|$

$a_n, a_{n-1} \geqslant \frac{1}{2}$

$$\therefore \frac{|a_{n-1} - a_n|}{(1+a_n)(1+a_{n-1})} \leqslant \frac{|a_n - a_{n-1}|}{\frac{3}{2} \cdot \frac{3}{2}} = \frac{4}{9} |a_n - a_{n-1}|$$

$$\therefore \quad |a_{n+1} - a_n| \leqslant \frac{4}{9} |a_n - a_{n-1}|$$

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} \frac{1}{1+a_{n-1}} = \frac{1}{1+\lim\limits_{n\to\infty} a_{n-1}}$$

$$= \frac{1}{1+L}$$

$$L = \frac{1}{1+L} \Rightarrow L^2 + L = 1 \Rightarrow L^2 + L - 1 = 0$$

$L = \frac{-1 \pm \sqrt{5}}{2}$ are possible solutions but we can discard $\frac{-1-\sqrt{5}}{2}$ since

$$\frac{1}{2} \leqslant a_n \leqslant 1$$

4.

i) Prove that the sequences $\{s_n\}$ and $\{t_n\}$ converge where $s_n = \sup\{x_k : k \geq n\}$ and $t_n = \inf\{x_k : k \geq n\}$.

Firstly we'll prove $\{s_n\}$ is non-increasing by contradiction

SFAC $\exists\, m, k \in \mathbb{N} : m < k \wedge S_m < S_k$

$\Rightarrow S_m \geqslant x_l : l \geqslant m \wedge S_k \geqslant x_l : l \geqslant k \quad (l \in \mathbb{N})$

Note $S_m \geqslant x_l : l \geqslant m \wedge m < k \Rightarrow S_m \geqslant x_l : l \geqslant k$

$\therefore$ S_m is an upper bound of $\{x_k : k \geqslant n\}$ but since $S_k = \sup\{x_k : k \geqslant n\} \wedge S_m < S_k$ we get a contradiction

$\therefore$ $\{S_n\}$ is non-increasing

$\inf\{x_n\} = I = t_1$ and $\sup\{x_n\} = S = s_1$

$I \leqslant S_n \leqslant S_1 \quad \forall\, n \in \mathbb{N}$ (trivial since $\sup\{S_n\} = S_1 = S$ and $I \leqslant x_n \ \forall\, n \in \mathbb{N}$)

Since S_n is bounded and non-increasing, by MCT

$\{S_n\}$ converges to $\inf\{S_n\}$

Note: Now that we have proved that $\{S_n\}$ converges we can similarly prove the same for $\{t_n\}$ by negating all x_n and showing the same or showing that $\{t_n\}$ is non-decreasing and bounded by $I = t_1 \leq t_n \leq S$

$\therefore$ By MCT it converges to $\sup\{t_n\}$.

ii) Find lim sup and lim inf of the sequence $a_n = \frac{3(-1)^n n^2}{n^2-n+1}$.

For $a_n = \dfrac{3 \cdot (-1)^n n^2}{n^2 - n + 1}$, $a_n < 0$ when n is odd and $a_n > 0$ when n is even

There is always another bigger n that happens to be odd/even. Since all a_n where n is even are bigger than all a_n where n is odd.

$\therefore$ $S_n \neq a_K$ if K is odd
$t_n \neq a_K$ if K is even

$$\lim_{n\to\infty} \sup\{a_k : k \geq n\} = \inf\{S_n\}$$

Since $\{S_n\}$ cannot be an odd, we can write it as

$$\lim_{n\to\infty} \sup\{a_{2K} : K \geq \tfrac{n}{2}, \; n, k \in \mathbb{N}\}$$

We can see that $\{S_n\}$ is decreasing.

$$\therefore \lim_{n\to\infty} \sup\{a_K : k \geq n\} = \lim_{n\to\infty} \frac{3 \cdot n^2}{n^2 - n + 1} \quad \left(\begin{array}{l}\text{Note } (-1)^n = 1 \\ \text{if } n \text{ is even}\end{array}\right)$$

We can see that this converges to 3
(trivial)

Similarly we can show that

$$\lim_{n\to\infty} \inf\{a_k : k \geqslant n\} = \lim_{n\to\infty} \frac{-3\cdot n^2}{n^2-n+1} = -3$$
(trivial)

$\therefore \lim_{n\to\infty} \sup = 3, \quad \lim_{n\to\infty} \inf = -3$

More generally

$\lim_{n\to\alpha} \sup = a_{2\lfloor\frac{\alpha+1}{2}\rfloor}$ and $\lim_{n\to\alpha} \inf = a_{2\lfloor\frac{\alpha+1}{2}\rfloor - 1}$

(trivial) even terms are decreasing, $\therefore$ sup $= a_n$ for smallest even n

odd terms are increasing, $\therefore$ inf $= a_n$ for (trivial) smallest odd n

iii) Show that if $\{a_n\}$ and $\{b_n\}$ are bounded sequences, then

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

Give an example in which two sides are not equal.

Let $\limsup a_n = \alpha_n$, $\limsup b_n = \beta_n$

By definition $\alpha_n \geqslant a_{k_1} \wedge \beta_n \geqslant b_{k_2} \ \forall \ k_1 \geqslant n$
$k_2 \geqslant n$

$\Rightarrow a_{k_1} + b_{k_2} \leqslant \alpha_n + \beta_n$

$\text{Lt sup}\,(a_n + b_n) = \text{Lt sup}\,\{a_k + b_k\,,\ k \geqslant n\}$

$a_k + b_k$ is a case of $a_{k_1} + b_{k_2}$ where $k_1 = k_2$

$\therefore\ a_{k_1} + b_{k_2} \leqslant \alpha_n + \beta_n \Rightarrow \text{Lt sup}\,(a_n + b_n) \leqslant \text{Lt sup}\,a_n + \text{Lt sup}\,b_n$

The equality of the two sides doesn't hold when one seq. is increasing and the other is decreasing.

For eg. $\{a_n = \frac{1}{n}\ \forall n \in \mathbb{N}\}$ and $\{b_n = -\frac{1}{n}\ \forall n \in \mathbb{N}\}$

$\{a_n + b_n = 0\ \forall n \in \mathbb{N}\} \Rightarrow \text{Lt sup}\,(a_n + b_n) = 0$

But $\underset{n \to \alpha}{\text{Lt}}\ \text{sup}\,a_n = a_\alpha\,,\quad \text{Lt sup}\,b_k = 0$

Their sum is always greater than 0 (trivial)
(unless $n \to \infty$)

5.

i) Prove that if the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

we don't care what it is, just that it exists

$\sum_{n=1}^{\infty} a_n$ converges $\therefore \lim_{k\to\infty} \sum_{n=1}^{k} a_n = S$

$$\sum_{n=1}^{k} a_n = \sum_{n=1}^{k-1} a_n + a_k$$

$$\lim_{k\to\infty} \sum_{n=1}^{k} a_n = S = \lim_{n\to\infty} \sum_{n=1}^{k-1} (a_n + a_k) \Rightarrow S = S + \lim_{n\to\infty} a_n$$

$$\therefore \lim_{n\to\infty} a_n = 0$$

ii) Determine whether the series $\sum_{n=1}^{\infty} \cos(\pi n)$ converges or diverges. If it converges, find the limit, if not, justify your answer.

$$\cos(\pi n) \begin{cases} 1, \text{ when } n \text{ is even} \\ -1 \text{ when } n \text{ is odd} \end{cases}$$

$(-1)^n$ has the same piecewise definition for $n \in \mathbb{N}$

$\therefore$ if we prove $\sum_{n=1}^{\infty} (-1)^n$ diverges that implies that $\sum_{n=1}^{\infty} \cos(\pi n)$ also diverges.

$$\sum_{n=1}^{\infty} \cos(\pi n) = \sum_{n=1}^{\infty} (-1)^n = S_n$$

When n is odd,

$$\sum_{n=1}^{2k+1} (-1)^n = (-1) + \sum_{n=1}^{k} (-1) + \sum_{n=1}^{k} (1)$$

$$= -1 + 0 = -1$$

When n is even

$$\sum_{n=1}^{2k} (-1)^n = \sum_{n=1}^{k} (-1) + \sum_{n=1}^{k} (1) = 0$$

$$\therefore \quad S_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

$\therefore \{S_k\}$ diverges $\rightarrow \sum_{n=1}^{\infty} (-1)^n$ diverges $\rightarrow \sum_{n=1}^{\infty} \cos(\pi n)$ diverges

iii) Let $r \in \mathbb{R}$ be such that $|r| < 1$. The series $\sum_{n=0}^{\infty} r^n$ is called the geometric series.

Prove that $S_n = \frac{1-r^{n+1}}{1-r}$, conclude that $\sum_{n=0}^{\infty} r^n$ converges and find the sum of the geometric series.

$$S_k = \sum_{n=0}^{k} r^n \quad , \quad S_k \cdot r = \sum_{n=0}^{k} r^{n+1} = \sum_{n=1}^{k+1} r^n$$

$$S_k \cdot r - S_k = \sum_{n=1}^{k+1} r^n + \sum_{n=0}^{k} r^n \Rightarrow S_k (r-1) = r^{k+1} - r^0$$

$$\Rightarrow \quad S_k = \frac{r^{k+1} - 1}{r-1} \cdot \frac{-1}{-1} = \frac{1 - r^{k+1}}{1-r}$$

We can show that $r < |1|$ implies that $\{S_n\}$ is contractive $\rightarrow$ Cauchy $\rightarrow$ converges

$$S_{k+1} - S_k = \sum_{n=0}^{k+1} r^n - \sum_{n=0}^{k} r^n = r^{k+1} = r \cdot r^k$$

$$S_k - S_{k-1} = \sum_{n=0}^{k} r^n - \sum_{n=0}^{k-1} r^n = r^k$$

$$\therefore \quad S_{k+1} - S_k = r \cdot (S_k - S_{k-1})$$

Taking abs value on both sides

$$|S_{k+1} - S_k| = |r(S_k - S_{k-1})| = |r| \cdot |S_k - S_{k-1}|$$

Since $-1 < r < 1$ ^ $|r| > 0 \;\forall n$

$$\Rightarrow \quad 0 < |r| < 1$$

Let $C = |r|$; $|S_{k+1} - S_k| \leqslant C|S_k - S_{k-1}|$

$\therefore$ $\{S_n\}$ is contractive $\rightarrow$ Cauchy $\rightarrow$ Converges

From (i) this implies $\lim\limits_{n\to\infty} r^n = 0$, $\lim\limits_{n\to\infty} r^{n+1} = 0$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1-r} = \frac{1 - \lim\limits_{n\to\infty} r^{n-1}}{1-r} = \frac{1-0}{1-r}$$

$$= \frac{1}{1-r} = S$$

6.

i) $\lim_{x\to 1} \frac{x^2-x+1}{x+1}$

First we'll try to plug in 1

we get $\frac{1^2-1+1}{1+1} = \frac{1}{2}$. Since this function is continuos at $x=1$, limit should be $\frac{1}{2}$

we will confirm using the ε-δ definition

Given $\varepsilon>0 \rightarrow$ P.T. $\exists\ \delta>0 : 0<|x-1|<\delta \Rightarrow |f(x)-L|<\varepsilon$

approaches 1

$\frac{1}{2}$ here

$$\left|f(x)-\frac{1}{2}\right| = \left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right| = \left|\frac{(x-1)(2x-1)}{(x+1)}\right|$$

$$= |x-1| \cdot \left|\frac{2x-1}{x+1}\right| = |x-1| \cdot \left|2-\frac{3}{x+1}\right|$$

Let's take $\delta = \frac{1}{2} \Rightarrow 0<x<1 \Rightarrow$

$$\Rightarrow 1<x+1<2 \Rightarrow \frac{1}{2}<\frac{1}{x+1}<1$$

$$\Rightarrow -3<\frac{-3}{x+1}<-\frac{3}{2}$$

$$\Rightarrow -1<\frac{-3}{(x+1)}+2<\frac{1}{2}$$

$$\Rightarrow \left|2-\frac{3}{(x+1)}\right|<1$$

$$\therefore \left| f(x) - \frac{1}{2} \right| = |x-1| \cdot \left| 2 - \frac{3}{(x+1)} \right| < |x-1| \cdot 1 = |x-1|$$

For $|x-1| < \varepsilon$, $\delta = \min\left(\frac{1}{2}, \varepsilon\right)$

if $0 < |x-1| < \delta$, then

$$\left| f(x) - \frac{1}{2} \right| \underset{\text{For } \delta \leq \frac{1}{2}}{<} |x-1| \underset{\text{For } \delta \leq \varepsilon}{<} \varepsilon$$

Hence $\lim\limits_{x \to 1} \dfrac{x^2 - x + 1}{x+1} = \dfrac{1}{2}$

ii) $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$

$f(x) = \sin\left(\frac{1}{x^2}\right)$

We will prove limit does not exist by seq. characterisation of Limits

We observe that $\forall\ n \in \mathbb{N},\ \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1$

and $\sin\left(\frac{3\pi}{2} + 2\pi n\right) = -1$

$x_n = \frac{1}{\sqrt{\frac{\pi}{2} + 2\pi n}}$, $y_n = \frac{1}{\sqrt{\frac{3\pi}{2} + 2\pi n}}$

$\sin\left(\frac{1}{x_n^2}\right) = 1 \quad \forall\ x_n$, $\sin\left(\frac{1}{y_n^2}\right) = -1 \quad \forall\ y_n$

As $n \to \infty$, $x_n, y_n \longrightarrow 0$ (trivial)

However as $f(x_n) = 1$, $f(y_n) = -1 \quad \forall n \in \mathbb{N}$

we get $f(x_n) \longrightarrow 1$ and $f(y_n) \longrightarrow -1$

This is suffecient to show that $f(x)$ diverges

$\underset{x\to 0}{Lt} \sin\left(\frac{1}{x^2}\right)$ DNE