My grades for **A1**

Q1

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Problem 1. (parts *(i)*, *(ii)*, and *(iii)*)

Problem 1

Part a

Base Case, we take two positive real numbers x_1, x_2

$(x_1 - x_2)^2 \geq 0 \implies x_1^2 + x_2^2 - 2x_1x_2 \geq 0$
$x_1^2 + x_2^2 \geq 4x_1x_2 \implies \frac{(x_1+x_2)^2}{4} \geq x_1x_2$
$\implies \frac{x_1+x_2}{2} \geq \sqrt{x_1x_2}$

We have proved it for any two positive real numbers x_1, x_2

We take 2 sets of n positive real numbers
$x_1, x_2, x_3, x_4 \dots x_n$ and $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4} \dots n_{2n}$
such that
$\frac{x_1, x_2, x_3, x_4 \dots x_n}{n} \geq \sqrt[n]{x_1x_2x_3x_4 \dots x_n}$
and
$\frac{x_{n+1}+x_{n+2}+x_{n+3}+x_{n+4}+\dots x_{2n}}{n} \geq \sqrt[n]{x_{n+1}x_{n+2}x_{n+3}x_{n+4} \dots x_{2n}}$
Adding these two inequalities we get

$\frac{x_1+x_2+x_3+x_4+\dots x_{2n}}{n} \geq \sqrt[n]{x_1x_2x_3 \dots x_n} + \sqrt[n]{x_{n+1}x_{n+2}x_{n+3} \dots x_{2n}}$

$$\frac{x_1 + x_2 + x_3 + x_4 + \dots x_{2n}}{2n} \geq \frac{\sqrt[n]{x_1x_2x_3 \dots x_n} + \sqrt[n]{x_{n+1}x_{n+2}x_{n+3} \dots x_{2n}}}{2} \tag{1}$$

Let $\sqrt[n]{x_1x_2x_3 \dots x_n}, \sqrt[n]{x_{n+1}x_{n+2}x_{n+3} \dots x_{2n}}$ be positive real numbers α, β respectively

$(\alpha - \beta)^2 \geq 0 \implies \alpha^2 + \beta^2 - 2\alpha\beta \geq 0$
$\alpha^2 + \beta^2 \geq 4\alpha\beta \implies \frac{(\alpha+\beta)^2}{4} \geq \alpha\beta$
$\implies \frac{\alpha+\beta}{2} \geq \sqrt{\alpha\beta}$

Therefore

$$\frac{\sqrt[n]{x_1x_2x_3 \dots x_n} + \sqrt[n]{x_{n+1}x_{n+2}x_{n+3} \dots x_{2n}}}{2} \geq \sqrt[2n]{x_1x_2x_3 \dots x_{2n}} \tag{2}$$

$$\implies \frac{x_1 + x_2 + x_3 + x_4 + \dots x_{2n}}{2n} \geq \sqrt[2n]{x_1x_2x_3 \dots x_{2n}} \tag{3}$$

Therefore using induction we can prove the AM GM inequality for $n = 2^k$ since we have proved base case of $n = 2$

Part b

We have proven that the AM GM inequality holds true for $n = 2^k$ numbers. If we can prove that the inequality being true for n numbers implies that it is true for $n-1$ numbers, we can fill in the gaps between exponents of two and show that it holds true for all numbers.

I worked on this backwards on paper and that's how I will be proving it here by using bi-implicative statements.

We need to prove that for any real numbers $x_1, x_2, x_3 \ldots x_{n-1}$

$$\frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1 x_2 x_3 \ldots x_{n-1}} \tag{4}$$

$$\iff \left(\frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1}\right)^{n-1} \geq x_1 x_2 x_3 \ldots x_{n-1} \tag{5}$$

If we multiply both sides of an inequality by a positive number they inequality holds and visa versa.
We multiply it by the positive number α where $\alpha = \frac{x_1+x_2+x_3+\ldots x_{n-1}}{n-1}$

$$\iff \left(\frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1}\right)^{n-1} \cdot \alpha \geq x_1 x_2 x_3 \ldots x_{n-1}\alpha \tag{7}$$

$$\iff \frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1} \geq \sqrt[n]{a_1 a_2 a_3 a_4 \ldots a_{n-1}\alpha} \tag{8}$$

We observe that (expansion has been left as exercise for TA Hint: Multiple numerator and denominator by n)
$\frac{x_1+x_2+x_3+\ldots x_{n-1}}{n-1} = \frac{x_1+x_2+x_3+\ldots x_{n-1}+\alpha}{n}$

$$\iff \frac{x_1 + x_2 + x_3 + \ldots x_{n-1} + \alpha}{n} \geq \sqrt[n]{a_1 a_2 a_3 a_4 \ldots a_{n-1}\alpha} \tag{9}$$

Since α is a positive real number, we can take a set of numbers $a_1, a_2, a_3 \ldots a_n$ where a_n is α. We know that the AM GM inequality holds for it since $n = 2^k$ and therefore the last statement is true. Since these statements are bi-implicative, it must also hold for the first $n-1$ numbers of this set or all numbers in this set other than $a_n(\alpha)$ confirming our initial statement:

$$\frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1 x_2 x_3 \ldots x_{n-1}} \tag{10}$$

to be true

Part c

Let's take a set of n positive real numbers $a_1, a_2, a_3 \ldots a_n$ where $a_i = i$

Because of the AM GM inequality we know that

$$\sqrt[n]{\prod a_i} \leq \frac{\sum a_i}{n} \tag{11}$$

In this case since $a_i = i$ we know that

$$\sqrt[n]{\prod_{i=1}^{n} i} \leq \frac{\sum_{i=1}^{n} i}{n} \tag{12}$$

$n! = \prod_{i=1}^{n} i$ (This is the definition of factorials)
$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (This is trivial)

Good work
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Therefore

$$\sqrt[n]{n!} \leq \frac{\frac{n(n+1)}{2}}{n} \implies \sqrt[n]{n!} \leq \frac{n+1}{2} \implies n! \leq \left(\frac{n+1}{2}\right)^n \tag{13}$$

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Problem 2. (parts *(i)*, *(ii)* and *(iii)*)

PROBLEM 2

i) $\frac{2}{5\sqrt[3]{n}+1} \leqslant \frac{2}{5\sqrt[3]{n}}$

and

$-\frac{2}{5\sqrt[3]{n}+1} \geqslant \frac{-2}{5\sqrt[3]{n}}$

Trivial

$\Rightarrow -\frac{2}{5\sqrt[3]{n}} \leqslant \frac{(-1)^n \cdot 2}{5\sqrt[3]{n}+1} \leqslant \frac{2}{5\sqrt[3]{n}} \quad \forall n \in \mathbb{N}$

By the squeeze theorem if

$\lim_{n\to\infty} \frac{-2}{5\sqrt[3]{n}} = 0 \quad \wedge \quad \lim_{n\to\infty} \frac{2}{5\sqrt[3]{n}} = 0$

$\Rightarrow \lim_{n\to\infty} \frac{(-1)^n \cdot 2}{5\sqrt[3]{n}+1} = 0$

To prove $\lim_{n\to\infty} \frac{-2}{5\sqrt[3]{n}}$

we need to prove that

$\forall \varepsilon > 0 \ \exists N$ s.t. $N \in \mathbb{N} \wedge$

$\left|\frac{-2}{5\sqrt[3]{n}} - 0\right| < \varepsilon \Rightarrow \left|\frac{-2}{5\sqrt[3]{n}}\right| < \varepsilon$

since $\frac{-2}{5\sqrt[3]{n}}$ is always negative

$\Rightarrow \frac{2}{5\sqrt[3]{n}} < \varepsilon \Leftarrow$

To prove $\lim_{n\to\infty} \frac{2}{5\sqrt[3]{n}}$

we need to prove that

$\forall \varepsilon > 0 \ \exists N$ s.t. $N \in \mathbb{N} \wedge$

$\left|\frac{2}{5\sqrt[3]{n}} - 0\right| < \varepsilon \Rightarrow \left|\frac{2}{5\sqrt[3]{n}}\right| < \varepsilon$

since $\frac{2}{5\sqrt[3]{n}}$ is always positive

$\frac{2}{5\sqrt[3]{N}} < \varepsilon \Rightarrow \frac{1}{\sqrt[3]{N}} < \frac{2}{5}\varepsilon \Rightarrow \sqrt[3]{N} > \frac{5}{2\varepsilon} \Rightarrow N > \frac{125}{8\cdot\varepsilon^3}$

Therefore taking any $N > \frac{125}{8\cdot\varepsilon^3}$ for example $N = \left\lfloor \frac{125}{8\cdot\varepsilon^3} \right\rfloor + 1$ works

$\therefore$ By squeeze theorem $\lim_{n\to\infty} \frac{(-1)^n \cdot 2}{5\sqrt[3]{n}+1} = 0$

$-\frac{1}{10} < \frac{(-1)^n \cdot 2}{5\sqrt[3]{n}+1} < \frac{1}{10} \Rightarrow \left| \frac{(-1)^n \cdot 2}{5\sqrt[3]{n}+1} \right| < 10^{-1}$

if $\varepsilon = 10^{-1}$, $N > \frac{125}{8\cdot\varepsilon^3} \Rightarrow N > \frac{125}{8\cdot(10^{-1})^3} =$

need
$N < 1000$
-1

Taking $N = 5^6 + 1$ or taking any $n \geq 15626$ satisfies this condition

ii) Let's assume for contradiction that $3^{\sqrt[3]{n}}$ converges and $\lim_{n\to\infty} 3^{\sqrt[3]{n}} = L$

$\forall\ \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $|3^{3\sqrt{n}} - L| < \varepsilon\ \forall\ n \geq N$

$\Rightarrow L - \varepsilon < 3^{3\sqrt{n}} < L + \varepsilon$

Looking at $3^{3\sqrt{n}} < L + \varepsilon \Rightarrow n < \left(\frac{\log_3(L+\varepsilon)}{3}\right)^2$

n will eventually exceed that no. and all values of n after that will do as well for a finite value of L

$n < n+1 \Rightarrow \sqrt[3]{n} < \sqrt[3]{n+1}$ $(n > 0)$

$\Rightarrow 3^{\sqrt[3]{n}} < 3^{\sqrt[3]{n+1}}$

since it is increasing the limit

$\lim_{n \to \infty} 3^{\sqrt[3]{n}} = +\infty$

iii) Let $n = (1+\alpha_n)^n \Rightarrow \sqrt[n]{n} = 1 + \alpha_n$

$n = (1+\alpha_n)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha_n^k = 1 + \sum_{k=1}^{n} \binom{n}{k} \alpha_n^k$

$\Rightarrow 0 \leqslant \binom{n}{k} \alpha_n^k \leqslant n-1 < n \quad \forall\ k \geqslant 1$

$\therefore 0 \leqslant \binom{n}{2} \alpha_n^2 < n$

$\binom{n}{2} \alpha_n^2 < n \Rightarrow \frac{n(n-1)}{2} \alpha_n^2 < n \Rightarrow \frac{(n-1)}{2} \alpha_n^2 < 1$

$0 \leqslant \alpha_n^2 < \frac{2}{(n-1)}$

$\lim_{n \to \infty} 0 = 0$, $\lim_{n \to \infty} \frac{2}{n-1} = 0$ (Trivial)

$\therefore$ By the Squeeze Theorem

$\lim_{n \to \infty} \alpha_n^2 = 0 \Rightarrow \lim_{n \to \infty} \alpha_n = 0$

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} (1+\alpha_n) = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \alpha_n$$

$$= 1 + 0 = 1$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{n} = 1$$

Q3

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Problem 3. (parts *(i)*, *(ii)* and *(iii)*)

PROBLEM 3

i) $\{a_n\}$ is convergent , $a_n \geqslant 0 \quad \forall\, n \in \mathbb{N}$

$\lim\limits_{n\to\infty} a_n = L$

For contradiction let's assume $L < 0$

$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N}$ s.t.

$|a_n - L| < \varepsilon \qquad \forall\, n \geqslant 0$

Since $a_n \geqslant 0$, $L < 0 \quad \forall\, n \in \mathbb{N}$, $|a_n - L| = a_n - L$

$a_n - L < \varepsilon \quad \Rightarrow \quad a_n < \varepsilon + L$

Setting $\varepsilon = -\frac{L}{2}$ $\left(L < 0 \Rightarrow \frac{L}{2} < 0 \Rightarrow -\frac{L}{2} > 0 \right)$

$a_n < \frac{L}{2}$ This is a contradiction since $a_n \geqslant 0$ and $L < 0$

$\therefore$ Our assumption that $L < 0$ is wrong and $L \geqslant 0$

ii) Since $a_n \geqslant 0 \;\forall n \Rightarrow \frac{a_{n+1}}{a_n} \geqslant 0 \;\forall n \Rightarrow L \geqslant 0$

$\alpha = \frac{L+1}{2} \quad \Rightarrow \quad 0 \leqslant L < \alpha < 1$

$\exists\, N \in \mathbb{N}$ s.t.

$L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon \qquad \forall\, n \geqslant N$

Let $\varepsilon = \alpha - L$

$\Rightarrow \quad -\alpha < \frac{a_{n+1}}{a_n} < \alpha$

$-\alpha < 0 \leqslant \frac{a_{n+1}}{a_n}$

$\frac{a_{n+1}}{a_n} < \alpha \quad \forall \quad n \geqslant N$

$\Rightarrow \quad a_{N+1} < \alpha \cdot a_N$

and $\frac{a_{N+2}}{a_{N+1}} < \alpha \quad \Rightarrow \quad a_{N+2} < \alpha \cdot a_{N+1} \Rightarrow a_{N+2} < a_N \alpha^2$

$$0 < a_n < a_N \alpha^{n-N}$$

$\lim_{n\to\infty} 0 = 0$

$\lim_{n\to\infty} a_N \alpha^{n-N}$

Since $\alpha < 1$

$\lim_{n\to\infty} \alpha^n = 0$

$\therefore \lim_{n\to\infty} a_N \alpha^{n-N} = 0$

$\therefore$ By Squeeze Theorem

$\lim_{n\to\infty} a_n = 0$

iii) $a_n = k \quad \forall n \in \mathbb{N}$, k is a constant (Trivial)

If $\lim_{n\to\infty} a_n \neq 0$ e.g. $a_n = \frac{3n+2}{5n+6}$

These are convergent sequences satisfying the above

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

$a_n = n+1$ is a divergent sequence that satisfies $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

Pf.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}$$

$$= 1 + 0 = 1$$

Perfect
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Q4

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Problem 4.

Problem 4

$\lim_{n\to\infty} a_n = L \Rightarrow \forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t.

$|a_n - L| < \varepsilon \quad \forall n \geq N \quad n \in \mathbb{N}$

$$L - \varepsilon < a_n < L + \varepsilon \quad \forall n \geq N$$

$$\frac{\sum_{i=1}^{N-1} a_i}{n} + \frac{(n-N)(L-\varepsilon)}{n} < \frac{\sum_{i=1}^{n} a_i}{n} < \frac{\sum_{i=1}^{N-1} a_i}{n} + \frac{(n-N)(L+\varepsilon)}{n}$$

Subtracting L from all sides

$$\frac{\sum_{i=1}^{N-1} a_i}{n} + \frac{-NL+N\varepsilon-n\varepsilon}{n} < \sum_{i=1}^{n} \frac{a_i}{n} - L < \frac{\sum_{i=1}^{N-1} a_i}{n} + \frac{-NL-N\varepsilon+n\varepsilon}{n}$$

Perfect 10

$$\left| \sum_{i=1}^{n} \frac{a_i}{n} - L - \left(\frac{\sum_{i=1}^{N-1} a_i}{n} - \frac{NL}{n} \right) \right| < \frac{(n-N)\varepsilon}{n}$$

Let $-\left(\frac{\sum_{i=1}^{N-1} a_i}{n} - \frac{NL}{n} \right) = \alpha$

$$\lim_{n\to\infty} -\left(\frac{\sum_{i=1}^{N-1} a_i - NL}{n} \right) = 0$$

This is trivial since $\sum_{i=1}^{N-1} a_i - NL$ is a constant w.r.t. n

$$\Rightarrow \lim_{n\to\infty} \alpha = 0$$

$$\left|\sum_{i=1}^{n} \frac{a_i}{n} + \alpha - L\right| < \frac{(n-N)}{n}\varepsilon \quad , \quad \varepsilon_0 = \frac{(n-N)}{n}\varepsilon$$

$$\forall \ \frac{(n-N)}{n}\varepsilon > 0 \quad \text{or} \quad \varepsilon_0 > 0$$

$$\forall \ \varepsilon_0 > 0 \qquad \exists \ N_0 \in \mathbb{N} :$$

$$\left|\sum_{i=1}^{n} \frac{a_i}{n} + \alpha - L\right| < \frac{(n-N)\varepsilon}{n} \qquad \forall \ n \geq N_0$$

$$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} \frac{a_i}{n} + \alpha = L$$

But since $\lim_{n\to\infty} \alpha = 0 \Rightarrow \lim_{n\to\infty} \sum \frac{a_i}{n} = L$

You can write it more formally. Choose a N_1 such that the first term can be made smaller than $\epsilon/2$ and $n \geq N_2 = N$ will make the second term smaller than $\epsilon/2$. Clubbing both of these while choosing $n \geq \max N_1, N_2$, you have the whole expression is less than ϵ.

Q5

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Problem 5. (parts *(i)*, *(ii)* and *(iii)*)

Problem 5

i) Let $\sup S = \beta$

$$\beta \geq s \qquad \forall s \in S$$

$$\Rightarrow \beta + a \geq a + s \qquad \forall s \in S$$

$$\text{or} \quad \beta + a \geq s_0 \qquad \forall s_0 \in a + S$$

$$\therefore \beta + a = \sup(a+S) = \sup S + a$$

You only showed that $a + \sup(S)$ is an upper bound for $a + S$. You need to show it is the least upper bound

-2

ii) If $a > 0 \Rightarrow (x, y \in \mathbb{R},\ x \leq y \Rightarrow ax \leq ay)$

$$\beta \geq s \qquad \forall s \in S$$

$$\Rightarrow a \cdot \beta \geq s \cdot a \qquad \text{since } a > 0$$

$$\text{or} \quad a \cdot \beta \geq s_0 \qquad \forall s_0 \in aS$$

$$\therefore a\beta = \sup(aS) = a \cdot \sup(S)$$

You only showed that $a \cdot \sup(S)$ is an upper bound for aS. You need to show it is the least upper bound

-2

iii) If $a < 0 \Rightarrow (x, y \in \mathbb{R},\ x \leq y \Rightarrow ax \geq ay)$

For eg. $a = -1$, $x, y = 2, 3$

$2 \leq 3$ but $-1 \cdot 2 \not\leq -1 \cdot 3$ or $-2 \not\leq -3$

$\beta \geq s \quad \forall s \in S$

$a\beta \leq as \quad$ if $\quad a < 0$

or $\quad a\beta \leq s_0 \quad \forall \; s_0 \in aS$

$\therefore \; a\beta = glb(aS) = a \cdot \sup(S)$

Q6

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Problem 3. (parts *(i)* and *(ii)*)

PROBLEM 6

i) $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \dots \frac{1}{n!}$

$S = \{a_1, a_2 \dots\} \subset \mathbb{R}$

To Prove $s \leqslant 3 \quad \forall s \in S$

Let $b_n = 1 + 2^0 + 2^{-1} + 2^{-2} \dots 2^{-n}$

$B = \{b_1, b_2 \dots\} \subset \mathbb{R}$

$\forall n \in \mathbb{N} \quad b_n \geqslant a_n$

We can verify this by induction. It is obvious for $n = 1, 2$

If $b_K \geqslant a_K \Rightarrow b_K + \frac{1}{(K+1)!} \geqslant a_K + \frac{1}{(K+1)!}$

$\frac{1}{(K+1)!} \leqslant \frac{1}{2^{K+1}} \quad \forall K \in \mathbb{N}, K \geqslant 2$ (Note $a > b \Rightarrow a \geqslant b$)

$b_K + \frac{1}{2^{K+1}} \geqslant a_K + \frac{1}{(K+1)!} \Rightarrow b_{K+1} \geqslant a_{K+1}$

$\therefore$ By induction $b_n \geqslant a_n \quad \forall n \in \mathbb{N}$

b_k is non decreasing since $b_{n+1} = b_n + \frac{1}{2^{n+1}}$

$\frac{1}{2^{n+1}} > 0 \quad \forall \quad n \in \mathbb{N}$

By the Monotone Convergence theorem

$\lim\limits_{n\to\infty} b_n = \text{lub}(B)$

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} 1 + 2^0 + 2^{-1} + 2^{-2} \cdots 2^{-n} = 2 + \lim\limits_{n\to\infty} 2^{-1} + 2^{-2} \cdots 2^{-n}$

$= 2 + \sum\limits_{k=1}^{\infty} \frac{1}{2^k} = 2+1 = 3$

$($=1 is trivial$)$

$\text{lub}(B) = 3 \Rightarrow b_0 \leqslant 3 \quad \forall \; b_0 \in B$

$\therefore \quad \forall n \in \mathbb{N} \qquad a_n \leqslant b_n \leqslant 3$

ii) $a_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2\cdots}}}}_{n \text{ times}}$

$\therefore \; a_1 = \sqrt{2}, \quad a_2 = \sqrt{2+\sqrt{2}} \quad \cdots$

we can define a_n recursively as

$a_n = \sqrt{2 + a_{n+1}}$

we know that a_n is increasing

$0 \leqslant a_n \leqslant a_{n+1} \leqslant 2$

Pf by Induction

We have proved base case above

$0 \leqslant \sqrt{2} \leqslant \sqrt{2+\sqrt{2}} \leqslant 2$

For some k let

$0 \leqslant a_k \leqslant a_{k+1} \leqslant 2$

$2 + a_k \leqslant 2 + a_{k+1} \leqslant 4$

$\sqrt{2+a_k} \leqslant \sqrt{2+a_{k+1}} \leqslant 2$

$\therefore$ By induction $a_n \leqslant a_{n+1} \leqslant 2 \quad \forall n \in \mathbb{N}$

By MCT $\{a_n\}$ converges, so $\exists L \in \mathbb{R}$ s.t. $\lim_{n\to\infty} a_n = L$

Let $\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_n^2 = L^2$

$L^2 = \lim_{n\to\infty} 2 + a_n \Rightarrow L^2 = 2 + L$

$L^2 - L - 2 = 0 \Rightarrow (L+1)(L-2) = 0$

Since $a_n > 0 \Rightarrow L > 0$

$L = 2$